Published by Institute of Physics Publishing for SISSA
Received: *September 14, 2005*
Accepted: *October 5, 2005*
Published: *October 20, 2005*

Heterotic geometry without isometries

Alexei P. Isaev and Osvaldo P. Santillan

Bogoliubov Laboratory of Theoretical Physics
JINR, 141 980 Dubna, Moscow Reg., Russia
E-mail: isaevap@thsun1.jinr.ru, osvaldo@thsun1.jinr.ru

Abstract: We present some properties of hyperkahler torsion (or heterotic) geometry in four dimensions that make it even more tractable than its hyperkahler counterpart. We show that in $d = 4$ hypercomplex structures and weak torsion hyperkahler geometries are the same. We present two equivalent formalisms describing such spaces, they are stated in the propositions of section 1. The first is reduced to solve a non-linear system for a doublet of potential functions, first found by Plebanski and Finley. The second is equivalent to finding the solutions of a quadratic Ashtekar-Jacobson-Smolin like system, but without a volume preserving condition. This is why heterotic spaces are simpler than usual hyperkahler ones. We also analyze the strong version of this geometry. Certain examples are presented, some of them are metrics of the Callan-Harvey-Strominger type and others are not. In the conclusion we discuss the benefits and disadvantages of both formulations in detail.

Keywords: Sigma Models, Superstrings and Heterotic Strings, Differential and Algebraic Geometry.

http://jhep.sissa.it/archive/papers/jhep102005061/jhep102005061.pdf

JHEP10(2005)061

Contents

1. Introduction

Supersymmetric σ models in two dimensions with $N = 2$ supersymmetry occur on Kähler manifolds while $N = 4$ occurs on hyperkahler spaces [1]. More general supersymmetric σ models can be constructed by including Wess-Zumino-Witten type couplings in the action [2]–[5]. These couplings can be interpreted as torsion potentials and the relevant geometry is a generalization of the Kähler and hyperkahler ones with closed torsion. Such spaces are known as strong Kähler and hyperkahler torsion (HKT) geometries. The first examples of $N = (4, 4)$ SUSY sigma model with torsion (and the corresponding HKT geometry) were found in [6] and developed further in [7]–[9] by using of the harmonic superspace formalism.

Hyperkähler torsion geometry is also an usefull mathematical tool in order to construct heterotic string models [10]–[11]. In particular heterotic $(4, 0)$ supersymmetric models are those that lead to strong hyperkahler torsion geometry [12]–[14]. If the torsion is not closed we have a weak hyperkahler geometry and this case has also physical significance [16]–[18].

A direct way of classifying the possible HKT spaces is to find the most general weak hyperkahler spaces and after that to impose the strong condition, i.e, the closure of the torsion. Callan, Harvey and Strominger noticed that under a conformal transformation any usual hyperkahler geometry is mapped into one with torsion [16]; such examples are sometimes called minimal in the literature [30] and they are not the most general [28]. As there is a particular interest in constructing spaces with at least one tri-holomorphic Killing vector due to applications related to dualities [19], there were found heterotic extensions of the known hyperkahler spaces and, in particular, the Eguchi-Hanson and Taub-Nut ones. The heterotic Eguchi Hanson geometry was shown to be conformal to the usual one, while heterotic Taub-Nut is a new geometry [28].

One of the main properties of hyperkahler torsion spaces is the integrability of the complex structures, that is, the annulation of their Nijenhuis tensor. In four dimensions this implies that the Weyl tensor of such manifolds is self-dual [20]–[22]. The converse of this statement is not true in general. On the other hand there exist a one to one correspondence between four dimensional self-dual structures with at least one isometry and 3-dimensional Einstein-Weyl structures [27]. The Einstein-Weyl condition is a generalization of the Einstein one to include conformal transformations, and weak hyperkahler spaces should correspond with certain special Einstein-Weyl metrics. In [29]–[30] it was shown that the self-dual spaces corresponding to the round three sphere and the Berger sphere (which are Einstein-Weyl) are of heterotic type. Arguments related to the harmonic superspace formalism suggest that indeed there are more examples [31]–[33].

The present work is related to the construction of weak heterotic geometries in $d = 4$ without Killing vectors. This problem is of interest also because any weak space with isometries should arise as subcases of those presented here. For the sake of clarity we resume the result presented in this letter in the following two equivalent propositions.

Proposition 1. *Consider a metric g defined on a manifold M together with three complex structures J^i satisfying the algebra $J^i \cdot J^j = -\delta_{ij} + \epsilon_{ijk} J^k$ and for which the metric is quaternion hermitean, i.e, $g(X,Y) = g(J^i X, J^i Y)$. Define the conformal family of metrics $[g]$ consisting of all the metrics g' related to g by an arbitrary conformal transformation.*

(a) Then we have the equivalence

$$d\overline{J}^i + \alpha \wedge \overline{J}^i = 0 \qquad \Longleftrightarrow \qquad N^i(X,Y) = 0\,, \tag{1.1}$$

where $\overline{J}^i$ and $N^i(X,Y)$ are the Kähler form and the Niejenhuis tensor associated to J^i, d is the usual exterior derivative and α is a 1-form. If any of (1.1) hold for g, then (1.1) is also satisfied for any g' of the conformal structure $[g]$, i.e, (1.1) is conformally invariant.

(b) Any four dimensional weak hyperkahler torsion metric is equivalent to one satisfying (1.1) and there exists a local coordinate system (x,y,p,q) for which the metric take the form

$$g = (dx - \Phi_x dp + \Phi_x dq) \otimes dp + (dy + \Psi_y dp - \Psi_x dq) \otimes dq\,, \tag{1.2}$$

up to a conformal transformation $g \to \omega^2 g$. The potentials Ψ and Φ satisfy the non-linear system

$$[\Phi_y \partial_x \partial_x + \Psi_x \partial_y \partial_y - (\Phi_x + \Psi_y)\partial_x \partial_y + \partial_x \partial_p + \partial_y \partial_q] \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = 0. \tag{1.3}$$

(c) Conversely any metric (1.2) defines a conformal family $[g]$ in which all the elements g' are weak hyperkahler torsion metrics. The torsion T corresponding to (1.2) is

given by

$$T = -\Xi_x dq \wedge (dy \wedge dx + \Phi_y dp \wedge dx - \Psi_y dy \wedge dp) + \\ + \Xi_y dp \wedge (dy \wedge dx + \Phi_x dy \wedge dq - \Psi_x dq \wedge dx)\,, \tag{1.4}$$

where $\Xi = \Phi_x - \Psi_y$. *Under the conformal transformation* $g \to \omega^2 g$ *the torsion is transformed as* $T \to T + *_g 2d\log(\omega)$.[1]

Proposition 1 should not be considered as a generalization of the Kähler formalism for weak HKT spaces. Although (Φ, Ψ) is a doublet potential, the metric (1.2) is not written in complex coordinates. The use of holomorphic coordinates for such spaces is described in detail in [28]. The following is an Ashtekar-Jacobson-Smolin like formulation for the same geometry.

Proposition 2. *Consider a representative* $g = \delta_{ab} e^a \otimes e^b$ *of a conformal family* $[g]$ *defined on a manifold* M *as at the beginning of proposition 1,* e^a *being tetrad 1-forms for which the metric is diagonal.*

(a) Then all the elements g' *of* $[g]$ *will be weak hyperkähler torsion iff*

$$\begin{aligned} [e_1, e_2] + [e_3, e_4] &= -A_2 e_1 + A_1 e_2 - A_4 e_3 + A_3 e_4 \\ [e_1, e_3] + [e_4, e_2] &= -A_3 e_1 + A_4 e_2 + A_1 e_3 - A_2 e_4 \\ [e_1, e_4] + [e_2, e_3] &= -A_4 e_1 - A_3 e_2 + A_2 e_3 + A_1 e_4 \end{aligned} \tag{1.5}$$

where e_a *is the dual tetrad of* e^a *and* A_i *are arbitrary functions on* M.

(b) Conversely any solution of (1.5) defines a conformal family $[g]$ *in which all the elements* g' *are weak hyperkahler torsion metrics. The torsion* T *corresponding to (1.2) is given by*

$$T = *_g (A_a - c^b_{ab}) e^a\,, \tag{1.6}$$

where c^c_{ab} *are the structure functions defined by the Lie bracket* $[e_a, e_b] = c^c_{ab} e_c$. *The transformation of the torsion under* $g \to \omega^2 g$ *follows directly from proposition 1.*

To conclude, we should mention that the properties of hyperkahler torsion geometry in higher dimension were considered for instance, in [46]–[52]. In particular, the quotient construction for HKT was achieved in [48]. Also it was found that when a sigma model is coupled to gravity the resulting target metric is a generalization of quaternionic Kähler geometry including torsion [50]. This result generalizes the classical one given by Witten and Bagger [53], who originally did not include a Wess-Zumino term to the action. A generalization of the Swann extension for quaternion Kähler torsion spaces was achieved [47].

[1] The action of the Hodge star $*_g$ is defined by

$$*_g e^a = \epsilon_{abcd} e^b \wedge e^c \wedge e^d\,.$$

To the knowledge of the authors the harmonic superspace description of quaternion Kähler geometry was already obtained in [54, 55] but the extension to the torsion case is still an open problem.

The present work is organized as follows. In section 2 we define what is weak and strong heterotic geometry. We show that the problem to finding weak examples is equivalent to solving a conformal extension of the hyperkahler condition (namely, the first (1.1) given above) together with the integrability condition for the complex structure. In section 3 we present the consequences of (1.1) following a work of Plebanski and Finley [22]. We found out that integrability and the conformal extension of the hyperkahler condition are equivalent in $d = 4$, which is the point (a) of proposition 1. Therefore weak heterotic geometry and hypercomplex structures are exactly the same concept. We also show that strong representatives of a given heterotic structure are determined by a conformal factor satisfying an inhomogeneous Laplace equation. Just in the case when the structure contains an hyperkahler metric it is possible to eliminate the inhomogeneous part by a conformal transformation. We discuss our results in the conclusions, together with possible applications. For completeness, we show in the appendix how this geometry arise in the context of supersymmetric sigma models.

2. Hyperkähler torsion manifolds

2.1 Main properties

In this section we define what is hyperkahler torsion geometry (for an explanation of its physical meaning see Appendix and references). We deal with $4n$-dimensional riemannian manifolds M with metric expressed as

$$g = \delta_{ab} e^a \otimes e^b \,. \tag{2.1}$$

Here e^a is a tetrad basis for which the metric is diagonal, and it is defined up to an SO$(4n)$ rotation. It is convenient to introduce the $4n \times 4n$ matrices

$$J^1 = \begin{pmatrix} 0 & -I_{n\times n} & 0 & 0 \\ I_{n\times n} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{n\times n} \\ 0 & 0 & I_{n\times n} & 0 \end{pmatrix}, \qquad J^2 = \begin{pmatrix} 0 & 0 & -I_{n\times n} & 0 \\ 0 & 0 & 0 & I_{n\times n} \\ I_{n\times n} & 0 & 0 & 0 \\ 0 & -I_{n\times n} & 0 & 0 \end{pmatrix}$$

$$J^3 = J^1 J^2 = \begin{pmatrix} 0 & 0 & 0 & -I_{n\times n} \\ 0 & 0 & -I_{n\times n} & 0 \\ 0 & I_{n\times n} & 0 & 0 \\ I_{n\times n} & 0 & 0 & 0 \end{pmatrix}. \tag{2.2}$$

Then it can be immediately be checked that (2.2) satisfy the multiplication rule of the quaternions

$$J^i \cdot J^j = -\delta_{ij} + \epsilon_{ijk} J^k \,. \tag{2.3}$$

In particular from (2.3) it is seen that $J^i \cdot J^i = -I$, a property that resembles the condition $i^2 = -1$ for the imaginary unity. The matrices (2.2) are of course not the only ones satisfying this property, in fact any simultaneous SO$(4n)$ rotation of J^i leaves the multiplication (2.3) unchanged. With the help of (2.2) we define the $(1,1)$ tensors

$$J^i = (J^i)^b{}_a e_b \otimes e^a \tag{2.4}$$

which are known as almost complex structures. Here e_a denote the dual of the 1-form e^a. Let us introduce the triplet $\overline{J}^i$ of $(0,2)$ tensors by

$$\overline{J}^i(X,Y) = g(X, J^i Y)\,. \tag{2.5}$$

One can easily see that the metric (2.1) is quaternion hermitean with respect to any of the complex structures (2.4), that is

$$g(J^i X, Y) = -g(X, J^i Y) \tag{2.6}$$

for any X,Y in $T_x M$ (in this notation $J^i X$ denotes the contraction of J^i with X). By virtue of (2.6) the tensors (2.5) are skew-symmetric and define locally a triplet of 2-forms which in the vielbein basis take the form

$$\overline{J}^i = (\overline{J}^i)_{ab} e^a \wedge e^b\,. \tag{2.7}$$

The 2-forms (2.7) are known as the hyperkahler triplet.

Definition 1. *Heterotic geometry (or torsion hyperkahler geometry) is defined by the following requirements.*

(1) Hypercomplex condition. *The almost complex structures (2.12) should be integrable. This mean that the Niejenhuis tensor*

$$N^i(X,Y) = [X,Y] + J^i[X, J^i Y] + J^i[J^i X, Y] - [J^i X, J^i Y] \tag{2.8}$$

associated with the complex structure J^i, is zero for every pair of vector fields X and Y in TM_x.

(2) Existence of torsion. *There exists a torsion tensor $T^{\mu}_{\nu\alpha}$ defined in terms of the metric for which the $T_{\mu\nu\alpha} = g_{\mu\xi} T^{\xi}_{\nu\alpha}$ is fully skew-symmetric and therefore define a three form (we write it in the tetrad basis)*

$$T = \frac{1}{3!} T_{\mu\nu\lambda} dx^{\mu} \wedge dx^{\nu} \wedge dx^{\lambda}\,. \tag{2.9}$$

Here greek indices denote the quantities related to a coordinate basis x^{μ}.[2] *If (2.9) is closed, i.e, $dT = 0$, the geometry will be called strong, otherwise it is called weak.*

[2]In the vienbein basis relation (2.9) will be

$$T = \frac{1}{3!} T_{abc} e^a \wedge e^b \wedge e^c\,.$$

(3) Covariant constancy of J^i. *Let us define a derivative D_μ with the connection*

$$\Upsilon^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} - \frac{1}{2}T^{\rho}_{\mu\nu}\,, \tag{2.10}$$

$\Gamma^{\rho}_{\mu\nu}$ being the Christofell symbols of the Levi-Civita connection. By definition the structures $(J^i)^b_a$ of hyperkahler torsion spaces satisfy

$$D_\mu (J^i)^\rho_\nu = 0 \tag{2.11}$$

that is, they are covariantly constant with respect to D_μ.

We are concerned through this work with $d=4$. In this case the explicit form of the $(1,1)$ tensors (2.4) is

$$\begin{aligned} J^1 &= -e_1\otimes e^2 + e_2\otimes e^1 - e_3\otimes e^4 + e_4\otimes e^3 \\ J^2 &= -e_1\otimes e^3 + e_3\otimes e^1 - e_4\otimes e^2 + e_2\otimes e^4 \\ J^3 &= -e_1\otimes e^4 + e_4\otimes e^1 - e_2\otimes e^3 + e_3\otimes e^2\,, \end{aligned} \tag{2.12}$$

and the action of (2.14) over the tangent space TM_x is defined by

$$\begin{array}{llll} J^1(e_1) = e_2\,, & J^1(e_2) = -e_1\,, & J^1(e_3) = e_4\,, & J^1(e_4) = -e_3\,, \\ J^2(e_1) = e_3\,, & J^2(e_2) = -e_4\,, & J^2(e_3) = -e_1\,, & J^2(e_4) = e_2\,, \\ J^3(e_1) = e_4\,, & J^3(e_2) = e_3\,, & J^3(e_3) = -e_2\,, & J^3(e_4) = -e_1\,. \end{array} \tag{2.13}$$

The hyperkahler triplet (2.7) is given by

$$\begin{aligned} \overline{J}^1 &= e^2\wedge e^1 + e^4\wedge e^3 \\ \overline{J}^2 &= e^3\wedge e^1 + e^2\wedge e^4 \\ \overline{J}^3 &= e^4\wedge e^1 + e^3\wedge e^2\,, \end{aligned} \tag{2.14}$$

By a discussion given above (2.2) with $n=1$ are a non unique 4×4 representation of the algebra (2.3) but defined up to an SO(4) rotation. It follows from (2.6) that the metric $g(X,Y)$ is quaternion hermitean with respect to any SO(4) rotated complex structures.

Condition $N^i(X,Y)=0$ for every J^i in (2.12) is equivalent to

$$\begin{aligned} [e_1,e_2]+[e_3,e_4] &= -A_2e_1 + A_1e_2 - A_4e_3 + A_3e_4 \\ [e_1,e_3]+[e_4,e_2] &= -A_3e_1 + A_4e_2 + A_1e_3 - A_2e_4 \\ [e_1,e_4]+[e_2,e_3] &= -A_4e_1 - A_3e_2 + A_2e_3 + A_1e_4 \end{aligned} \tag{2.15}$$

with certain functions A_i whose form depends on the metric in consideration [20]. Equations (2.15) holds by simply evaluating $N^i(e_a,e_b)=0$ with the use of (2.8) and (2.13), the calculation is straightforward and we omit it. The system (2.15) is invariant under a transformation

$$e_a \to \omega e_a\ (g\to\omega^2 g)\,, \qquad A_a \to A_a + e_a\log\omega\,.$$

Therefore the integrability condition is conformal invariant, that is, any metric g with integrable complex structures define a conformal structure $[g]$ with the same property. The family $[g]$ is called hypercomplex structure. Hypercomplex condition implies, but is not implied by, that the Weyl tensor of $[g]$ is self-dual [21]. In the limit $A_i = 0$ (2.15) reduces to a system equivalent to the Ashtekar-Jacobson-Smolin one [26] for hyperkahler spaces. Nevertheless one can not conclude that (2.15) always describes spaces which are conformally equivalent to hyperkahler ones, even in the case $A_a = 0$. [3] In the special cases in which the resulting family is conformal hyperkahler, we will say that is of the Callan-Harvey-Strominger type [16].

It is important to recall that although the complex structures are defined up to certain SO(4) automorphisms of (2.3), this does not affect the integrability condition $N^i(X,Y) = 0$. A SO(4) rotation of the complex structures can be compensated by an SO(4) rotation of the frame e^a leaving (2.13) and therefore $N^a(e_i, e_j) = 0$ invariant. By the results to be presented below it will be clear that such rotations also do not affect conditions 2 and 3.

2.2 Relation with the Plebanski-Finley conformal structures

Requirements 1, 2 and 3 of the definition of HKT geometry are not independent because the last two imply the first. To see why is it so, let us write the expression for the Nijenhuis tensor in a coordinate basis x^μ,

$$N^\rho_{\mu\nu} = (J^i)^\lambda_\mu[\partial_\lambda(J^i)^\rho_\nu \;-\; \partial_\nu(J^i)^\rho_\lambda] \;-(\mu \;\leftrightarrow\; \nu) = 0\,. \tag{2.16}$$

Then (2.16) together with (2.11) implies that

$$T_{\rho\mu\nu} - (J^i)^\lambda_{[\mu}(J^i)^\sigma_\nu T_{\rho]\lambda\sigma} = 0\,. \tag{2.17}$$

It is simple to check that (2.17) is satisfied for any skew-symmetric torsion. Let us consider the tetrad basis e^a for which the complex structures take the form (2.2), then (2.17) is a direct consequence of the skew symmetry of $T_{\mu\nu\alpha}$. Since $N^i(X,Y)$ is a tensor, if is zero in one basis then it is zero in any basis. Thus we have proved that the requirements 2 and 3 imply first. We will reach to the same conclusion in the next section.

We will show now that the task to find a weak HKT geometry is in fact equivalent to obtaining the solutions of

$$d\overline{J}^i + \alpha\wedge\overline{J}^i = 0\,, \tag{2.18}$$

where α is an arbitrary 1-form. To prove this we note that the relation

$$2D_\mu g_{\nu\alpha} = -g_{\mu\xi}T^\xi_{\nu\alpha} - g_{\nu\xi}T^\xi_{\mu\alpha}$$

together with the skew symmetry of $T_{\mu\nu\alpha} = g_{\mu\xi}T^\xi_{\nu\alpha}$ yields directly that

$$D_\mu g_{\nu\alpha} = 0\,. \tag{2.19}$$

[3]In order to obtain an hyperkahler manifold the tetrad should also preserve certain volume form. If this condition holds the structures are called minimal.

Equation (2.19) implies the equivalence

$$D_\mu (J^i)^\nu_\alpha = 0 \qquad \Longleftrightarrow \qquad D_\mu (\overline{J}^i)_{\nu\alpha} = 0\,, \tag{2.20}$$

and therefore, for any hyperkahler torsion space

$$D_\mu (\overline{J}^i)_{\nu\alpha} = 0\,.$$

From the identity [24]

$$D_\mu (\overline{J}^i)_{\nu\alpha} = D_{[\mu} (\overline{J}^i)_{\nu\alpha]} - 3D_{[\mu} (\overline{J}^i)_{\xi\rho]} (J^i)^\xi_\nu (J^i)^\rho_\alpha + (\overline{J}^i)_{\mu\xi} (N^i)^\xi_{\nu\alpha}\,,$$

it is easily seen that

$$D_\mu (\overline{J}^i)_{\nu\alpha} = 0 \qquad \Longleftrightarrow \qquad D_{[\mu} (\overline{J}^i)_{\nu\alpha]} = 0\,, \qquad N^i(X,Y) = 0\,. \tag{2.21}$$

where the brackets denote the totally antisymmetric combination of indices. However we have seen that $N^i(X,Y) = 0$ is satisfied by our hypothesis. A direct calculation using (2.11), (2.10) and the skew-symmetry of $T_{\mu\nu\alpha}$ shows that

$$D_{[\mu} (\overline{J}^i)_{\nu\alpha]} = 0 \qquad \Longleftrightarrow \qquad d\overline{J}^i + \frac{1}{2} T_{dbc} (J^i)^d_a e^a \wedge e^b \wedge e^c = 0\,. \tag{2.22}$$

The second (2.22) is the generalization of the hyperkahler condition for torsion manifolds. If the torsion $T_{\mu\nu\alpha}$ is zero, then the second (2.22) implies that the hyperkahler triplet is closed, which is a well known feature of hyperkahler geometry. Consider now the explicit form of (2.22) for $\overline{J}^1$. Then (2.11) and the first (2.2) gives us that

$$\begin{aligned} &d\overline{J}^1 + T_{234} e^1 \wedge e^3 \wedge e^4 - T_{134} e^2 \wedge e^3 \wedge e^4 + T_{312} e^4 \wedge e^1 \wedge e^2 - T_{412} e^3 \wedge e^1 \wedge e^2 = \\ &= d\overline{J}^1 + (T_{234} e^1 + T_{134} e^2 - T_{123} e^4 + T_{124} e^3) \wedge (e^1 \wedge e^2 + e^3 \wedge e^4) = 0 \end{aligned} \tag{2.23}$$

and therefore

$$d\overline{J}^1 + (*_g T) \wedge \overline{J}^1 = 0\,.$$

The same is obtained for $\overline{J}^2$ and $\overline{J}^3$, namely

$$d\overline{J}^i + (*_g T) \wedge \overline{J}^i = 0\,. \tag{2.24}$$

Let us define now the 1-form α given by

$$\alpha = *_g T \qquad \Longleftrightarrow \qquad *_g \alpha = T\,. \tag{2.25}$$

Then we have from (2.24) that (2.22) is equivalent to

$$d\overline{J}^i + \alpha \wedge \overline{J}^i = 0\,. \tag{2.26}$$

Therefore formulae (2.20)–(2.22) implies that hyperkahler torsion geometry is completely characterized by (2.18) which is what we wanted to show.

Condition (2.26) is the conformally invariant extension of the hyperkahler one and was studied by Plebanski and Finley in the seventies [22]. A detailed calculation shows that (2.26) is invariant under

$$e^a \to \omega e^a \; (g \to \omega^2 g) \qquad \alpha \longrightarrow \alpha + 2d\log\omega\,, \tag{2.27}$$

and, in consequence, it defines a conformal family $[g]$. The transformation law (2.27) implies in particular that if α is a gradient there exist a representative of $[g]$ for which

$$\alpha = 0 \qquad \Longleftrightarrow \qquad d\overline{J}^i = 0\,.$$

Clearly such families have an hyperkahler element and are of the Callan-Harvey-Strominger type. The main result of Plebanski and Finley is that in general (2.26) implies that the Weyl tensor of $[g]$ is self-dual [22]. From (2.25) we deduce that under $g \to \omega^2 g$ it follows that

$$T \longrightarrow T + *_g 2d\log(\omega)\,, \tag{2.28}$$

which is a transformation found by Callan et all in [16].

Some more comments are in order. As we have seen, requirements 2 and 3 imply that $N^i(X,Y) = 0$ and also imply (2.26). Therefore

$$N^i(X,Y) = 0 \qquad \Longleftarrow \qquad d\overline{J}^i + \alpha\wedge\overline{J}^i = 0\,. \tag{2.29}$$

However it is also known the implicature [20]–[21]

$$N^i(X,Y) = 0 \qquad \Longrightarrow \qquad d\overline{J}^i + \alpha\wedge\overline{J}^i = 0\,. \tag{2.30}$$

To see how (2.30) holds consider the connection ω given by the first structure equation

$$de^a + \omega^a_b \wedge e^b = 0\,.$$

It is well known that the antisymmetric part $\omega^a_{[bc]}$ is related to the structure functions defined by the Lie bracket $[e_a, e_b] = c^c_{ab} e_c$ by

$$\omega^a_{[bc]} = \frac{1}{2} c^a_{bc}\,.$$

Consider now the form

$$\alpha = A - \chi\,,$$

where

$$A = A_a e^a\,, \qquad \chi = c^b_{ab} e^a\,, \tag{2.31}$$

and the self-dual form $\overline{J}^1 = e^1\wedge e^2 + e^3\wedge e^4$. Then by use of (2.15) one obtain that

$$\begin{aligned} d\overline{J}^1 &= d(e^1\wedge e^2 + e^3\wedge e^4) \\ &= -\frac{1}{2} c^{[1}_{ab} e^{2]} \wedge e^a\wedge e^b - \frac{1}{2} c^{[3}_{ab} e^{4]} \wedge e^a \wedge e^b \\ &= e^1\wedge e^2 \wedge (c^a_{ab} e^b + A_3 e^3 + A_4 e^4) + e^3\wedge e^4\wedge (c^a_{ab} e^b + A_1 e^1 + A_2 e^2) \\ &= (e^1\wedge e^2 + e^3\wedge e^4)\wedge (A - \chi) \end{aligned} \tag{2.32}$$

and therefore

$$d\overline{J}^1 + \overline{J}^1 \wedge (\chi - A) = 0\,.$$

This is equivalent to (3.8) with $\alpha = \chi - A$. The same formula holds for $\overline{J}^2$ and $\overline{J}^3$. This means that we have shown the equivalence

$$N^i(X,Y) = 0 \qquad \Longleftrightarrow \qquad d\overline{J}^i + \alpha \wedge \overline{J}^i = 0\,, \tag{2.33}$$

that is, hypercomplex structures are *the same* as the conformal structures defined by (2.26). This result refers just to $d = 4$.

3. The general solution

We have seen in the previous section that to constructing an hyperkahler torsion space is equivalent to finding a metric which solves the conditions stated in (2.18). This task was solved by Plebanski and Finley and it is stated in the following proposition [22].

Proposition 3. *For any four dimensional metric g for which*

$$d\overline{J}^i + \alpha \wedge \overline{J}^i = 0$$

hold there exists a local coordinate system (x, y, p, q) for which g takes the form

$$g = (dx - \Phi_x dp + \Phi_x dq) \otimes dp + (dy + \Psi_y dp - \Psi_x dq) \otimes dq\,, \tag{3.1}$$

up to a conformal transformation $g \to \omega^2 g$. The functions Ψ and Φ depend on (x, y, p, q) and satisfy the non linear system

$$[\Phi_y \partial_x \partial_x + \Psi_x \partial_y \partial_y - (\Phi_x + \Psi_y)\partial_x \partial_y + \partial_x \partial_p + \partial_y \partial_q] \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = 0\,. \tag{3.2}$$

The converse of this assertion is also true.

For completeness we give the proof of Proposition 3 in some detail. It is convenient to introduce the Penrose notation and write the metric g in the following complex form

$$g = \delta_{ab} e^a \otimes e^b = E^1 \otimes E^2 + E^3 \otimes E^4\,. \tag{3.3}$$

We have defined the new complex tetrad

$$E^1 = \frac{1}{\sqrt{2}}(e^1 + ie^2)\,, \qquad E^2 = \frac{1}{\sqrt{2}}(e^1 - ie^2)$$
$$E^3 = \frac{1}{\sqrt{2}}(e^3 + ie^4)\,, \qquad E^4 = \frac{1}{\sqrt{2}}(e^3 - ie^4) \tag{3.4}$$

In terms of this basis the complex structures (2.14) are expressed as

$$\begin{aligned} \overline{J}^1 &= 2E^4 \wedge E^1, \qquad \overline{J}^3 = 2E^3 \wedge E^2 \\ \overline{J}^2 &= -E^1 \wedge E^2 + E^3 \wedge E^4\,. \end{aligned} \tag{3.5}$$

The advantage of this notation is that allows us to directly apply the Frobenius theorem.

Theorem 1. Frobenius theorem *In n-dimensions, if in a certain domain U there exist r 1-forms β^i $(i=1,\dots,r)$ such that*

$$\Omega = \beta^1 \wedge \cdots \wedge \beta^r \neq 0$$

and there exists a 1-form γ such that

$$d\Omega = \gamma \wedge \Omega \,, \tag{3.6}$$

then there exist some functions f^i_j and g^j on U such that

$$\beta^i = \sum_{j=1}^{r} f^i_j dg^j \,. \tag{3.7}$$

The two conditions

$$d\overline{J}^1 + \alpha \wedge \overline{J}^1 = 0\,, \qquad d\overline{J}^3 + \alpha \wedge \overline{J}^3 = 0 \tag{3.8}$$

of (2.26) are of the form (3.6) with E^i playing the role of the 1-forms β^i, $\overline{J}^1$ and $\overline{J}^3$ are the analogs of Ω and α play the role of γ. Then Frobenius theorem implies the existence of scalar functions $\widetilde{A},\dots,\widetilde{H}$ and $p,\dots,s$ such that

$$\begin{aligned} E^1 &= \widetilde{A}dp + \widetilde{B}dq\,, \\ E^2 &= \widetilde{E}dr + \widetilde{F}ds\,, \\ E^3 &= -\widetilde{G}dr - \widetilde{H}ds \\ E^4 &= -\widetilde{C}dp - \widetilde{D}dq\,. \end{aligned} \tag{3.9}$$

The functions $\widetilde{A},\dots,\widetilde{H}$ and $p,..,s$ are the analogs of f^i_j and g^j in (3.7), respectively.

Consider the two functions ϕ and f defined by

$$\begin{aligned} \widetilde{A}\widetilde{D} - \widetilde{B}\widetilde{C} &= (\phi^{-1}e^f)^2\,, \\ \widetilde{E}\widetilde{H} - \widetilde{F}\widetilde{G} &= (\phi^{-1}e^{-f})^2\,. \end{aligned} \tag{3.10}$$

It is more natural to use the variables

$$\begin{aligned} (A,B,C,D) &= (\phi^{-1}e^f)^{-1}(\widetilde{A},\widetilde{B},\widetilde{C},\widetilde{D}) \\ (E,F,G,H) &= (\phi^{-1}e^{-f})^{-1}(\widetilde{E},\widetilde{F},\widetilde{G},\widetilde{H}) \end{aligned} \tag{3.11}$$

with the functions $A,\dots,H$ normalized as

$$AD - BC = 1, \qquad EH - GF = 1. \tag{3.12}$$

Using (3.10) we can express (3.9) as

$$\begin{aligned} E^1 &= \phi^{-1}e^f(Adp + Bdq)\,, \\ E^2 &= \phi^{-1}e^{-f}(Edr + Fds)\,, \\ E^3 &= -\phi^{-1}e^{-f}(Gdr + Hds)\,, \\ E^4 &= -\phi^{-1}e^f(Cdp + Ddq)\,. \end{aligned} \tag{3.13}$$

Since

$$E^1 \wedge E^2 \wedge E^3 \wedge E^4 = \phi^{-4} dp \wedge dq \wedge dr \wedge ds \neq 0$$

one can consider the functions p, q, r and s as independent coordinates.

From (3.13), (3.5) and (3.8) it follows the expression for α

$$\alpha = d\log(\phi) + f_p dp + f_q dq - f_r dr - f_s ds\,. \tag{3.14}$$

We now consider the remaining condition (2.26), namely

$$d\overline{J}^2 + \alpha \wedge \overline{J}^2 = 0\,. \tag{3.15}$$

By the use of (3.5) together with (3.13) we find that

$$\begin{aligned} \overline{J}^2 = \phi^{-4}[&(AE+CG)dp \wedge dr + (AF+CH)dp \wedge ds + (BE+DG)dq \wedge dr + \\ &+ (BF+DH)dq \wedge ds]\,. \end{aligned} \tag{3.16}$$

A direct calculation shows that (3.15) together with (3.14) implies

$$\begin{aligned} (AE+CG)2f_s - (AF+CH)2f_r &= (AE+CG)_s - (AF+CH)_r\,, \\ (BE+DG)2f_s - (BF+DH)2f_r &= (BE+DG)_s - (BF+DH)_r\,, \\ (AE+CG)2f_q - (BE+DG)2f_p &= -(AE+CG)_q + (BE+DG)_p\,, \\ (AF+CH)2f_q - (BF+DH)2f_p &= -(AF+CH)_q + (BF+DH)_p\,. \end{aligned} \tag{3.17}$$

The first two (3.17) show that there exist certain functions x and y such that

$$\begin{aligned} AE+CG &= e^{2f}x_r\,, \\ BE+DG &= e^{2f}y_r\,, \\ AF+CH &= e^{2f}x_s\,, \\ BF+DH &= e^{2f}y_s\,. \end{aligned} \tag{3.18}$$

By multiplying the last two (3.17) by e^{2f} and using (3.18) we also obtain that

$$(e^{4f}x_r)_q = (e^{4f}y_r)_p\,, \qquad (e^{4f}x_s)_q = (e^{4f}y_s)_p\,. \tag{3.19}$$

The solution of (3.18) and (3.19) are

$$\begin{aligned} G &= A(e^{2f}y_r) - B(e^{2f}x_r) \\ H &= A(e^{2f}y_s) - B(e^{2f}x_s) \\ E &= D(e^{2f}x_r) - C(e^{2f}y_r) \\ F &= D(e^{2f}x_s) - C(e^{2f}y_s)\,. \end{aligned} \tag{3.20}$$

The normalization (3.12) and (3.20) implies that

$$J = \left| \begin{pmatrix} x_r & x_s \\ y_r & y_s \end{pmatrix} \right| = e^{-4f} \neq 0\,,$$

and therefore x and y can be used as independent coordinates instead of r and s. Moreover, the functions $A,\dots,D$ are not determined by the equations but are just constrained by the first (3.12). This reflects that the tetrad and the vielbein are defined up to an SO(4) rotation which leaves the condition $AD-BC=1$ invariant. Therefore we can select $A=D=1$ and $B=C=0$ withouth losing generality. The tetrad becames simplified as

$$\begin{aligned}
E^1 &= (\phi e^{-f})^{-1}dp\,,\\
E^2 &= (\phi e^{-f})^{-1}(dx+Kdp+Ldq)\,,\\
E^3 &= -(\phi e^{-f})^{-1}(dy+Mdp+Ndq)\,,\\
E^4 &= -(\phi e^{-f})^{-1}dq\,,
\end{aligned}\tag{3.21}$$

being $K=-x_p$, $L=-x_q$, $M=-y_p$ and $N=-y_q$. The dual basis of (3.21) is

$$\begin{aligned}
E_2 &= \phi e^{-f}\partial_x\\
E_1 &= \phi e^{-f}(\partial_p - K\partial_x - M\partial_y)\,,\\
E_4 &= -\phi e^{-f}(\partial_q - L\partial_x - N\partial_y)\,,\\
E_3 &= -\phi e^{-f}\partial_y\,.
\end{aligned}\tag{3.22}$$

Since our goal is to find a conformal structure, it is convenient to eliminate the factor ϕe^{-f} in (3.22) by a conformal transformation. Then from (3.8) it is calculated that

$$2\alpha = (L-M)_x dq - (L-M)_y dp\,.\tag{3.23}$$

Also, it follows from (3.15) and (3.23) that

$$\begin{aligned}
\partial_x K + \partial_y L = 0\,,\qquad \partial_x M + \partial_y N &= 0\,,\\
(\partial_p - K\partial_x - M\partial_y)L - (\partial_q - L\partial_x - N\partial_y)K &= 0\,,\\
(\partial_p - K\partial_x - M\partial_y)N - (\partial_q - L\partial_x - N\partial_y)M &= 0\,.
\end{aligned}\tag{3.24}$$

The first two (3.24) imply the existence of two functions Φ and Ψ such that

$$K=-\Phi_x\,,\qquad L=\Phi_x\,,\qquad M=\Psi_y\,,\qquad N=-\Psi_x\,,\tag{3.25}$$

and the last two (3.24) together with (3.25) give the following non-linear equations for the doublet (Φ,Ψ)

$$[\Phi_y\partial_x\partial_x + \Psi_x\partial_y\partial_y - (\Phi_x+\Psi_y)\partial_x\partial_y + \partial_x\partial_p + \partial_y\partial_q]\begin{pmatrix}\Phi\\ \Psi\end{pmatrix} = 0\,.\tag{3.26}$$

We conclude that the most general structure satisfying (2.26) is given in terms of two key functions Φ and Ψ satisfying (3.26). A simple calculation shows that the metric corresponding to the tetrad (3.22) is (3.1), which is what we wanted to prove (Q.E.D).

It will be instructive if we show that for any structure of the proposition 3 it follows that

$$N^i(X,Y)=0\,,$$

in accordance with (2.29). This is completely equivalent to solve the system (2.15), that is

$$\begin{aligned}
[e_1, e_2] + [e_3, e_4] &= -A_2 e_1 + A_1 e_2 - A_4 e_3 + A_3 e_4\,, \\
[e_1, e_3] + [e_4, e_2] &= -A_3 e_1 + A_4 e_2 + A_1 e_3 - A_2 e_4\,, \\
[e_1, e_4] + [e_2, e_3] &= -A_4 e_1 - A_3 e_2 + A_2 e_3 + A_1 e_4\,,
\end{aligned} \tag{3.27}$$

for the basis e_i corresponding to (3.1). From (3.4), (3.21) and (3.25) we find that

$$\begin{aligned}
e^1 &= \frac{1}{\sqrt{2}}[dp + (dx - \Phi_y dp + \Phi_x dq)]\,, \\
e^2 &= \frac{1}{i\sqrt{2}}[dp - (dx - \Phi_y dp + \Phi_x dq)] \\
e^3 &= -\frac{1}{\sqrt{2}}[dq + (dy + \Psi_y dp - \Psi_x dq)]\,, \\
e^4 &= \frac{1}{i\sqrt{2}}[dq - (dy + \Psi_y dp - \Psi_x dq)]\,.
\end{aligned} \tag{3.28}$$

The dual basis corresponding to (3.28) is

$$\begin{aligned}
e_1 &= \frac{1}{\sqrt{2}}[(\partial_p + \Phi_y \partial_x - \Psi_y \partial_y) + \partial_x]\,, \\
e_2 &= -\frac{1}{i\sqrt{2}}[(\partial_p + \Phi_y \partial_x - \Psi_y \partial_y) - \partial_x] \\
e_3 &= -\frac{1}{\sqrt{2}}[\partial_y + (\partial_q - \Phi_x \partial_x + \Psi_x \partial_y)]\,, \\
e_4 &= \frac{1}{i\sqrt{2}}[\partial_y - (\partial_q - \Phi_x \partial_x + \Psi_x \partial_y)]\,.
\end{aligned} \tag{3.29}$$

It is simple to check that the first (2.15) for (3.29) imply that $A_i = 0$. After some tedious calculation one obtain from the full system (2.15) that

$$[\Phi_y \partial_x \partial_x + \Psi_x \partial_y \partial_y - (\Phi_x + \Psi_y)\partial_x \partial_y + \partial_x \partial_p + \partial_y \partial_q] \begin{pmatrix} \Phi \\ \Psi \end{pmatrix} = 0\,,$$

which is (3.26). *This again shows that the equivalence (2.33) is true.*

We conclude that the general form of an HKT metric is

$$g = (dx - \Phi_y dp + \Phi_x dq) \otimes dp + (dy + \Psi_y dp - \Psi_x dq) \otimes dq$$

up to a conformal transformation. The form α in (3.23) can be expressed as

$$\alpha = \Xi_x E^1 - \Xi_y E^4, \tag{3.30}$$

where $\Xi = \Phi_x - \Psi_y$. One can prove by using (3.4) that

$$*\,E^1 = -E^2 \wedge E^3 \wedge E^4, \qquad *E^4 = -E^1 \wedge E^2 \wedge E^3. \tag{3.31}$$

Formula (3.31) together with (2.25) and (3.30) gives explicitly the torsion, namely

$$T = -\Xi_x E^2 \wedge E^3 \wedge E^4 + \Xi_y E^1 \wedge E^2 \wedge E^3. \tag{3.32}$$

All the results of this section are stated in the propositions of the introduction. It is instructive at this point to consider some simple examples of hyperkahler torsion spaces.

JHEP10(2005)061

Example 1. Let us construct weak heterotic spaces with two commuting Killing vectors. The coordinates will be named $(\rho, \eta, \theta, \varphi)$. Consider four functions $f_1, \ldots, f_4$ and $g_1, \ldots, g_4$ depending on the coordinates $x^1 = \rho$ and $x^2 = \eta$ and the vector fields

$$
\begin{aligned}
e_j &= f_j \partial_\theta + g_j \partial_\varphi + \partial_{x^j} \,, \qquad (j = 1, 2) \\
e_j &= f_j \partial_\theta + g_j \partial_\varphi \qquad (j = 3, 4) \,.
\end{aligned}
$$

The basis e_j is the most general for a metric with two commuting isometries up to a conformal change. Introducing this expressions into (2.15) gives $A_i = 0$ together with the system

$$
\begin{aligned}
(f_3)_\eta - (f_4)_\rho &= 0 \,, \qquad & (g_3)_\eta - (g_4)_\rho &= 0 \\
(f_2)_\eta + (f_1)_\rho &= 0 \,, \qquad & (g_2)_\eta + (g_1)_\rho &= 0 \\
(f_1)_\eta - (f_2)_\rho &= 0 \,, \qquad & (g_1)_\eta - (g_2)_\rho &= 0
\end{aligned}
\tag{3.33}
$$

and from the first we see that $f_3 = (H)_\rho$ and $f_4 = (H)_\eta$ for some function $H(\rho, \eta)$. This mean that we can make the coordinate change $(\rho, \eta, \theta, \varphi) \to (\rho - H, \eta, \theta, \varphi)$ and eliminate f_3 and f_4. The same holds for g_3 and g_4. From system (3.33) together with these simplifications we get Cauchy Riemann equations implying that $f(z) = e_1 - ie_2$ and $g(z) = f_1 - if_2$ are holomorphic functions depending on the argument $z = \rho + i\eta$. The corresponding weak heterotic metric is

$$
\widetilde{g} = d\rho^2 + d\eta^2 + \frac{(e_1 d\theta - f_1 d\varphi)^2 + (e_2 d\theta - f_2 d\varphi)^2}{(e_1 f_2 - e_2 f_1)^2} . \tag{3.34}
$$

The commuting Killing vectors are ∂_θ and ∂_φ.

Example 2. Another family of hypercomplex structures that can be constructed in terms of holomorphic functions. It can be directly checked that the system (2.15) with the coefficients $A_i = 0$ can be cast in the following complex form

$$
[e_1 + ie_2, e_1 - ie_2] - [e_3 + ie_4, e_3 - ie_4] = 0, \quad [e_1 + ie_2, e_3 - ie_4] = 0 \tag{3.35}
$$

Let M be a complex surface with holomorphic coordinates (z_1, z_2) and let us define four vector fields e_i as

$$
\begin{aligned}
e_1 + ie_2 &= f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} \\
e_3 + ie_4 &= f_3 \frac{\partial}{\partial z_1} + f_4 \frac{\partial}{\partial z_2}
\end{aligned}
\tag{3.36}
$$

being f_j a complex function on M. Then (3.35) implies that $\partial f_j / \partial \overline{z}_k = 0$ and therefore we can construct a family of hypercomplex structures by using four arbitrary holomorphic functions or two holomorphic vector fields [45].

Example 3. If we set $\Xi = \Phi_x - \Psi_y = 0$ in (3.30) it follows that there exist a function Θ such that

$$
\Phi = \Theta_y \,, \qquad \Psi = \Theta_x \,.
$$

JHEP10(2005)061

Then (3.26) implies that

$$\Theta_{xx}\Theta_{yy} - (\Theta_{xy})^2 + \Theta_{xp} + \Theta_{yq} = 0\,. \tag{3.37}$$

Equation (3.37) is known as the second heavenly equation and the corresponding metric

$$g = dp \otimes (dx - \Theta_{yy}dp + \Theta_{xy}dq) + dq \otimes (dy + \Theta_{xy}dp - \Theta_{xx}dq) \tag{3.38}$$

is the most general hyperkahler one [39]. There exists a coordinate system (r, s, p, q) defined in terms of (x, y, p, q) for which the metric (3.38) became

$$g = K_{pr}dp \otimes dr + K_{ps}dp \otimes ds + K_{qr}dq \otimes dr + K_{qs}dq \otimes ds\,. \tag{3.39}$$

The function K plays the role of a Kähler potential and satisfies the first heavenly equation

$$\begin{vmatrix} K_{pr} & K_{ps} \\ K_{qr} & K_{qs} \end{vmatrix} = 1. \tag{3.40}$$

It is clear that the weak families corresponding to (3.40) or (3.38) are those of the Callan et.all type. Particular solutions of (3.26) have been found for instance in [40], but the general solution is unknown even for (3.40).

The geometries presented till now are weak heterotic. Due to conformal invariance all the elements of the conformal family $[g]$ corresponding to (3.1) are also weak. Therefore in order to find an strong geometry we should not impose $dT = 0$ for the torsion (1.4) corresponding to (3.1) but instead, we must solve $d\widetilde{T} = 0$ where $\widetilde{T} = T + *_g 2d\log(\omega)$ is the torsion corresponding to the representative $\widetilde{g} = \omega^2 g$. After some calculation we obtain the following *linear* equation defining ω

$$\Delta_g \log(\omega) = \partial_{[1}T_{234]}\,, \tag{3.41}$$

where Δ_g is the Laplace operator for (3.1). The left hand side (3.41) does not depend on ω and acts as an inhomogeneous source. Then the spaces $\widetilde{g} = \omega^2 g$ will be a strong representatives of the family if ω solve (3.41). The spaces (3.1) will be strong only in the case that constant ω is a solution of the resulting equations, which implies that

$$\partial_{[1}T_{234]} = \Xi_{xp} - (\Xi_x\Phi_y)_y + (\Xi_x\Psi_y)_x + \Xi_{yq} - (\Xi_y\Phi_x)_x + (\Xi_y\Psi_x)_y = 0\,. \tag{3.42}$$

It also follows from the result of this section that if we deal with a minimal weak structure, then (3.41) is reduced to

$$\Delta_{\widehat{g}} \log(\omega) = 0\,, \tag{3.43}$$

being $\widehat{g}$ the hyperkahler representative of the family [16].

4. Conclusions

In the present work we attacked the problem of constructing weak and strong 4-dimensional hyperkahler torsion geometries without isometries. We have shown that the most general local form for a weak metric is defined by two potential functions satisfying a non linear system, up to a conformal transformation.

JHEP10(2005)061

We argue that the present work has practical applications, although the system of equations for the potentials is highly non-linear and difficult to solve. We showed that in four dimensions hypercomplex structures and weak torsion hyperkahler geometries are totally equivalent concepts. Concretely speaking, we showed that integrability of the complex structures implies and is implied by the other weak HKT properties. Therefore the problem to find weak heterotic geometries is reduced to find the solutions of an Ashtekar-Jacobson-Smolin like system without the volume preserving condition, which has the advantage to be quadratic.

From this discussion and the Ashtekar formulation of hyperkahler spaces it is clear that a weak HKT metric will be of the Callan-Harvey-Strominger type (i.e. conformal to an hyperkahler one) if there is a volume form preserved by its tetrad. Nevertheless not all the hypercomplex structures preserves a 4-form, the metrics given in [28]–[30] are counterexamples. We have seen also that a space is of Callan-Harvey-Strominger type if the torsion is the dual of a gradient. Therefore this condition and the volume preserving one should be equivalent.

The strong condition (i.e. the closure of the torsion form) is not invariant under a general conformal transformation and thus it does not define a conformal structure. Nevertheless the problem to find the strong representatives of a weak heterotic family $[g]$ is reduced to solving a *linear* non homogeneous laplacian equation over an arbitrary element g. The homogeneous part can be eliminated if the metric is conformal to a hyperkahler one.

At first sight it looks that the "Ashtekar like" formulation for HKT is the most convenient formulation. Instead we suggest that the other variant could be more useful in the context dualities, in which the presence of isometries is needed [19]. To make this suggestion more concrete, let us recall that self-dual spaces are described in terms of a Kähler potential satisfying the heavenly equation [39]. Although this equation is non-linear and the general solution is not known, the analysis of the Killing equations and the possible isometries allowed to classify the hyperkahler metrics with one Killing vector [38]. In particular it was found that they are divided in two types, the first described for by a monopole equation and the second by some limit of a Toda system [35]–[38]. We propose that it is worthy to generalize, if is possible, the mathematical machinery to classify the Killing equations of gravitational instantons [38]–[44] to the hyperkahler torsion case. Perhaps it will be possible to find the classification of heterotic geometry with one or more Killing vectors and in particular, to identify the subcases that match for duality applications. This goal has not been achieved yet, and our suggestion could be an alternative point of view of the references [30]–[33].

Acknowledgments

We are grateful to G.Valent for orientation about literature on the subject. O.P.S thanks specially to Luis Masperi for his permanent and unconditional support during his mandate at the CLAF (Centro Latinoamericano de Fisica).

A. Supersymmetric sigma models and HKT manifolds

In this section we sketch why strong torsion hyperkahler geometry is the target space metric of $N = 4$ supersymmetric sigma models [1]. Sigma models in dimension two are described by lagrangians of the form

$$I_b = -\frac{1}{2\pi}\int d^2x \left[g_{AB}(\varphi)\partial_\mu\varphi^A\partial^\mu\varphi^B + b_{AB}(\varphi)\epsilon^{\mu\nu}\partial_\mu\varphi^A\partial^\nu\varphi^B\right], \tag{A.1}$$

being g_{AB} an arbitrary metric tensor.[4] The fields φ^A are bosonic ($A = 1, \ldots, d$) and depend on the coordinates x^i ($i = 1, 2$). This means that φ^A parameterize two-cycles on a d-dimensional target manifold M with a metric g_{AB}. The second term is a Wess-Zumino-Witten type term.

If we want to make the supersymmetric extension of (A.1) we need to include a number of fermion degrees equal to the number of the boson ones. Thus, we need to introduce a set of spinors ψ'^A and impose a supersymmetry transformation, which on general grounds should be of the form

$$\delta\varphi^A = \overline{\varepsilon}_1 J^A_B\psi'^B, \tag{A.2}$$

where J is some $(1, 1)$ tensor. We still have the freedom to make the redefinition $\psi'^B = (J^{-1})^B_A\psi^A$ in (A.2), for which (A.2) is converted into

$$\delta\varphi^A = \overline{\varepsilon}_1\psi^A. \tag{A.3}$$

The $N = (1, 0)$ supersymmetric extension of (A.1) is [14]

$$I_{sb} = -\frac{1}{2\pi}\int d^2x \left[g_{AB}(\varphi)\partial_\mu\varphi^A\partial^\mu\varphi^B + b_{AB}(\varphi)\epsilon^{\mu\nu}\partial_\mu\varphi^A\partial^\nu\varphi^B + \overline{\psi}^A\gamma^\mu D_\mu\psi^B g_{AB}\right], \tag{A.4}$$

where the spinors ψ^A are right-handed and ε is left-handed. The covariant derivative D_μ is defined by the action

$$\begin{aligned} D_\mu\psi^A &= \partial_\mu\psi^A + \Upsilon^A_{BC}\psi^B\partial_\mu\varphi^C, \\ \Upsilon^A_{BC} &= \Gamma^A_{BC} + \frac{1}{2}T^A_{BC}, \end{aligned} \tag{A.5}$$

where Γ^A_{BC} are the Christofell symbols of the Levi-Civita connection for g_{AB}. The torsion form T_{ABC} is entirely defined in terms of the Wess-Zumino-Witten potential b_{AB} as

$$2T_{ABC} = -3b_{[AB,C]} \qquad \Longleftrightarrow \qquad dT = 0. \tag{A.6}$$

The indices A, B are lowered and raised through g_{AB}. Action (A.4) is invariant under (A.3) and

$$\delta\psi^A = \gamma^\mu\partial_\mu\varphi^A\varepsilon - \psi^B\overline{\varepsilon}S^A_{BC}\psi^C. \tag{A.7}$$

[4] In the main text we have used latin indices (a,b,...) as flat indices and greek indices (α, β,...) as curved ones. Instead in this appendix capital latin indices (A,B,...) are curved, we use capital latin indices and not greek in order to not confuse them with the greek indices used to denote space-time derivative ∂_μ or gamma matrices γ^μ.

JHEP10(2005)061

The tensor S_{ABC} is constrained by supersymmetry arguments to be skew symmetric and covariantly constant

$$S_{ABC;D} = S_{ABC,D} - 3S_{E[AB}\Upsilon^E_{C]D} = 0\,.$$

The presence of a covariantly constant 3-form gives a new restriction on the manifold. On group manifolds this condition has solution and S can be chosen proportional to T [14]. The commutator of two supersymmetries on the boson field is

$$[\delta,\delta']\varphi^A = \bar{\varepsilon}'\gamma^\mu\varepsilon\left(2\partial_\mu\varphi^A - \overline{\psi}^B\gamma_\mu S^A_{BC}\psi^C\right), \tag{A.8}$$

and corresponds to the usual supersymmetry algebra without the presence of central charges.

Let us consider now the problem of constructing an $N=(2,0)$ supersymmetric extension of (A.1). For this purpose we should impose a new supersymmetry transformation to (A.4) without spoiling (A.3) and (A.7). As we saw the general supersymmetry variation for φ^A is of the form

$$\delta^1\varphi^A = \bar{\varepsilon}_1(J^1)^A_B\psi^B\,, \tag{A.9}$$

being J^1 certain $(1,1)$ tensor. Due to the invariance of (A.4) with respect to the symmetry (A.7) we have to impose

$$I_{sb}(\varphi, J^1\psi) = I_{sb}(\varphi,\psi)\,. \tag{A.10}$$

Formula (A.10) implies that

$$g_{AB}(J^1)^B_C + (J^1)^B_A g_{BC} = 0, \tag{A.11}$$

and then the metric is hermitean with respect to the tensor J^1. Condition (A.10) also implies that

$$D_A(J^1)^B_C = 0 \qquad \Longrightarrow \qquad (J^1)_{[AB,C]} = -2(J^1)^D_{[A}T_{B,C]D}\,. \tag{A.12}$$

This means that J^1 is covariantly constant with respect to the connection with torsion. The compatibility condition of (A.12) is

$$(J^1)^A_E R^E_{BCD} = R^A_{ECD}(J^1)^E_B\,, \tag{A.13}$$

and more generally the tensor J^1 commute with all the generators of the holonomy group. The new supersymmetry transformations result

$$\delta^1\varphi^A = \bar{\varepsilon}_1(J^1)^A_B\psi^B \tag{A.14}$$

$$\delta^1\psi^A = \gamma^\mu\partial_\mu\varphi^B(J^1)^A_B\varepsilon_1 - [(J^1)^A_D\Upsilon^D_{BE}(J^1)^E_C + (S_1)^A_{BC}]\psi^B\bar{\varepsilon}_1\psi^C\,. \tag{A.15}$$

The tensor $(S_-)^A_{BC}$ is not determined by other quantities. The commutator of the two supersymmetries acting over φ^A is

$$\left[\delta,\delta^1\right]\varphi^A = \bar{\varepsilon}_1\gamma_\mu\varepsilon_1\left[(J^1)^A_B + (J^1)^B_A\right]\partial_\mu\varphi^B + \bar{\varepsilon}_1\gamma_\mu\varepsilon_1\overline{\psi}^B\gamma_\mu\psi^C(J^1)^A_D N^D_{BC}\,, \tag{A.16}$$

plus terms depending on $(S_1)^A_{BC}$. Then the usual supersymmetric algebra holds if (A.16) is zero and this implies that

$$(J^1)^A_B + (J^1)^B_A = 0\,, \qquad N^C_{AB} = 0\,, \qquad (S_1)^A_{BC} = 0\,. \tag{A.17}$$

The first equation (A.17) together with (A.11) implies that J^1 is an almost complex structure and therefore the dimension of the target manifold should be even (d=$2n$). The second (A.17) show that J^1 is integrable. The metrics g_{AB} for which (A.12) and (A.17) hold are Kähler torsion. We conclude that $N = 2$ supersymmetric sigma models occur on *Kähler torsion manifolds.*

If there is a third supersymmetry corresponding to another complex structure J^2 and to a parameter ε_2 then it is obvious that the previous reasoning is true and the properties (A.11), (A.12) and (A.17) hold for J^2. But we obtain new restrictions by requiring that the transformation corresponding to ε_1 and ε_2 close to a supersymmetry. The algebra

$$[\delta(\varepsilon_1), \delta(\varepsilon_2)]\varphi^A = 2\overline{\varepsilon}^i_2\gamma^\mu\varepsilon^j_1\delta_{ij}\partial_\mu\varphi^A \tag{A.18}$$

(which correspond to (A.8) with $S^A_{BC} = 0$) is obtained if and only if

$$\begin{aligned} &J^i \cdot J^j + J^j \cdot J^i = \delta_{ij} I \\ &N^{ij}(X, Y) = [X, Y] + J^i[X, J^j Y] + J^i[J^j X, Y] - [J^i X, J^j Y] = 0\,. \end{aligned} \tag{A.19}$$

$N^{ij}(X, Y)$ is known as the mixed Niejenhuis tensor.

Let us define the tensor $J^3 = J^1 \cdot J^2$ for a $N = (3, 0)$ supersymmetric sigma model. It follows from (A.19) that

$$(J^3)^2 = -I$$

and therefore J^3 is also an almost complex structure. It is direct to check

$$J^i \cdot J^j = -\delta_{ij} + \epsilon^{ijk} J^k\,. \tag{A.20}$$

The integrability of J^1 and J^2 implies the integrability of J^3 and (A.19). Also (A.12) for J^1 and J^2 imply that

$$D_A(J^3)^B_C = 0\,. \tag{A.21}$$

Therefore $N = (3, 0)$ supersymmetry implies $N = (4, 0)$ supersymmetry for sigma models and from (A.19), (A.6), (A.12) and (A.20) it follows that the target metric g_{AB} of a $N = (4, 0)$ supersymmetric sigma model is always *strong hyperkahler torsion.* If the form b_{AB} is zero, then the geometry presented here reduces to the usual Kähler and hyperkahler ones.

The weak cases are also of interest in the context of low energy heterotic string actions because they contain a three form that is not closed due to a Green-Schwarz anomaly. For more details of this assertions see [16]–[17].

References

[1] L. Alvarez-Gaume and D. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric sigma model*, *Commun. Math. Phys.* **80** (1981) 443.

[2] E. Witten, *Nonabelian bosonization in two-dimensions*, *Commun. Math. Phys.* **92** (1984) 455.

[3] T.L. Curtright and C.K. Zachos, *Geometry, topology and supersymmetry in nonlinear models*, *Phys. Rev. Lett.* **53** (1984) 1799.

[4] J. Gates, S. J., C.M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear sigma models*, *Nucl. Phys.* **B 248** (1984) 157.

[5] P.S. Howe and G. Sierra, *Two-dimensional supersymmetric nonlinear sigma models with torsion*, *Phys. Lett.* **B 148** (1984) 451.

[6] E. Ivanov and S. Krivonos, $N = 4$ *super-Liouville equation (In russian)*, *J. Phys.* **A 17** (1984) L671; *Bäcklund transformations for the super-Liouville equations*, *Theor. Math. Phys.* **63** (1985) 260

[7] E.A. Ivanov, S.O. Krivonos and V.M. Leviant, *A new class of superconformal sigma models with the Wess-Zumino action*, *Nucl. Phys.* **B 304** (1988) 601;
Quantum $N = 3$, $N = 4$ *superconformal WZW sigma models*, *Phys. Lett.* **B 215** (1988) 689, erratum *ibid.* **221** (1989) 432.

[8] O.E. Gorovoi and E.A. Ivanov, *Superfield actions for* $N = 4$ *WZNW-Liouville systems*, *Nucl. Phys.* **B 381** (1992) 394.

[9] E. Ivanov and A. Sutulin, *Sigma models in (4,4) harmonic superspace*, *Nucl. Phys.* **B 432** (1994) 246 [`hep-th/9404098`].

[10] A. Sen, *The heterotic string in arbitrary background field*, *Phys. Rev.* **D 32** (1985) 2102.

[11] C.M. Hull and E. Witten, *Supersymmetric sigma models and the heterotic string*, *Phys. Lett.* **B 160** (1985) 398.

[12] E. Bergshoeff and E. Sezgin, *The* $(4, 0)$ *heterotic string with Wess-Zumino term*, *Mod. Phys. Lett.* **A 1** (1986) 191.

[13] P. Howe and G. Papadopoulos, *Anomalies in two-dimensional supersymmetric nonlinear sigma models*, *Class. and Quant. Grav.* **4** (1987) 1794; *Further remarks on the geometry of two-dimensional nonlinear sigma models*, *Class. and Quant. Grav.* **5** (1988) 501.

[14] P. Spindel, A. Sevrin, W. Troost and A. Van Proeyen, *Extended supersymmetric sigma models on group manifolds, 1. The complex structures*, *Nucl. Phys.* **B 308** (1988) 662.

[15] C.G. Callan Jr., *Instantons and solitons in heterotic string theory*, `hep-th/9109052`.

[16] C.G. Callan Jr., J.A. Harvey and A. Strominger, *World sheet approach to heterotic instantons and solitons*, *Nucl. Phys.* **B 359** (1991) 611.

[17] A. Strominger, *Superstrings with torsion*, *Nucl. Phys.* **B 274** (1986) 253.

[18] G.W. Gibbons, G. Papadopoulos and K.S. Stelle, *HKT and OKT geometries on soliton black hole moduli spaces*, *Nucl. Phys.* **B 508** (1997) 623 [`hep-th/9706207`].

[19] A. Giveon and M. Roček, *Introduction to duality*, `hep-th/9406178`.

[20] J. Grant and I. Strachan, *Hypercomplex integrable systems*, `solv-int/9808019`.

[21] C.P. Boyer, *A note on hyperhermitian four-manifolds*, in *Proceedings of the American Mathematical Society* **102** (1988) 157.

[22] J. Plebanski and Finley, *Further heavenly metrics and their symmetries*, *J. Math. Phys.* **17** (1976) 585.

[23] N. Hitchin, *Complex manifolds and Einstein's equations in twistor geometry and non-linear systems*, H.D. Doebner and T.D. Palev eds., Springer 1982.

[24] A. Futaki, *Kähler-Einstein metrics and integral invariants*, *Lec. Notes Math. 1314*, Springer Verlag, Berlin, 1980.

[25] M. Atiyah, N. Hitchin and I. Singer *Self-duality in four dimensional riemannian geometry*, *Proc. R. Soc. London* **A 362** (1978) 425.

[26] A. Ashtekar T. Jacobson and L. Smolin, *A new characterization of half flat solutions to Einstein's equation*, *Commun. Math. Phys.* **115** (1988) 631;
L.J. Mason and E.T. Newman, *A connection between the Einstein and Yang-Mills equations*, *Commun. Math. Phys.* **121** (1989) 659.

[27] P. Jones and K. Tod, *Minitwistor spaces and Einstein-Weyl spaces*, *Class. and Quant. Grav.* **2** (1985) 565.

[28] G. Bonneau and G. Valent, *Local heterotic geometry in holomorphic coordinates*, *Class. and Quant. Grav.* **11** (1994) 1133 [hep-th/9401003].

[29] G. Papadopoulos, *Elliptic monopoles and (4,0) supersymmetric sigma models with torsion*, *Phys. Lett.* **B 356** (1995) 249 [hep-th/9505119].

[30] T. Chave, G. Valent and K.P. Tod, *(4,0) and (4,4) sigma models with a triholomorphic killing vector*, *Phys. Lett.* **B 383** (1996) 262.

[31] E.A. Ivanov, *Off-shell (4,4) supersymmetric sigma models with torsion as gauge theories in harmonic superspace*, *Phys. Lett.* **B 356** (1995) 239 [hep-th/9504070].

[32] E.A. Ivanov, *On the harmonic superspace geometry of (4,4) supersymmetric sigma models with torsion*, *Phys. Rev.* **D 53** (1996) 2201 [hep-th/9502073].

[33] F. Delduc, S. Kalitsyn and E. Sokatchev, *Geometry of sigma models with heterotic sypersymmetry*, *Class. and Quant. Grav.* **7** (1990) 1567.

[34] P.S. Howe and G. Papadopoulos, *Holonomy groups and W symmetries*, *Commun. Math. Phys.* **151** (1993) 467

[35] G.W. Gibbons and S.W. Hawking, *Gravitational multi-instantons*, *Phys. Lett.* **B 78** (1978) 430.

[36] S. Kloster, M. Som and A. Das, *On the stationary gravitational fields*, *J. Math. Phys.* **23** (1974) 1096.

[37] J. Gegenberg and A. Das, *Stationary riemanian space-times with self-dual curvature*, *Gen. Rel. Grav.* **16** (1982) 817

[38] C.P. Boyer and I. Finley, J. D., *Killing vectors in selfdual, euclidean Einstein spaces*, *J. Math. Phys.* **23** (1982) 1126.

[39] J.F. Plebanski, *Some solutions of complex Einstein equations*, *J. Math. Phys.* **16** (1975) 2395.

[40] M. Dunajski, *The twisted photon associated to hyper-hermitian four-manifolds*, *J. Geom. Phys.* **30** (1999) 266 [math.DG/9808137].

[41] M. Przanoskwi, *Locally Hermite Einstein, selfdual gravitational instantons*, *Acta Phys. Polon.* **B14** (1983) 625.

[42] M. Przanoskwi, *Lie-Bäcklund transformation and gravitational instantons*, *Acta Phys. Polon.* **B18** (1987) 879.

[43] M. Przanoskwi, *Delta isobar kählerian manifolds and selfdual gravitational instantons*, *Phys. Lett.* **A 110** (1985) 295

[44] M. Przanowski, *Killing vector fields in selfdual, euclidean Einstein spaces with lambda not equal 0*, *J. Math. Phys.* **32** (1991) 1004.

[45] D. Joyce Duke, *Explicit construction of self-dual 4-manifolds*, *Math. J.* **77** (1995) 519.

[46] G. Gantcharov and Y.S. Poon, *Geometry of hyper-Kähler connections with torsion*, *Commun. Math. Phys.* **213** (2000) 19

[47] Y.S. Poon and A. Swann, *Superconformal symmetry and hyper-Kähler manifolds with torsion*, *Commun. Math. Phys.* **241** (2003) 177

[48] G. Grantcharov, G. Papadopoulos and Y.S. Poon, *Reduction of hkt-structures*, *J. Math. Phys.* **43** (2002) 3766 [math.dg/0201159].

[49] L. Ornea, Y.S. Poon and A. Swann, *Potential one-forms for hyperkähler structures with torsion*, *Class. and Quant. Grav.* **20** (2003) 1845 [math.dg/0211427].

[50] A. Opfermann and G. Papadopoulos, *Homogeneous HKT and QKT manifolds*, math-ph/9807026.

[51] P.S. Howe and G. Papadopoulos, *Twistor spaces for HKT manifolds*, *Phys. Lett.* **B 379** (1996) 80 [hep-th/9602108].

[52] A. Opfermann, *T-duality and HKT manifolds*, *Phys. Lett.* **B 416** (1998) 101 [hep-th/9709048].

[53] J. Bagger and E. Witten, *Matter couplings in $N = 2$ supergravity*, *Nucl. Phys.* **B 222** (1983) 1.

[54] E. Ivanov and G. Valent, *Quaternionic metrics from harmonic superspace: lagrangian approach and quotient construction*, *Nucl. Phys.* **B 576** (2000) 543 [hep-th/0001165].

[55] A. Galperin, E. Ivanov and O. Ogievetsky, *Harmonic space and quaternionic manifolds*, *Ann. Phys. (NY)* **230** (1994) 201;
A. Galperin and O. Ogievetsky, *Harmonic potentials for quaternionic symmetric sigma models*, *Phys. Lett.* **B 30** (1993) 67.